Indian Statistical Institute, Bangalore

M. Math.I Year, First Semester

Semestral Examination

Analysis of Several Variables

Time: 3 hours December 4th, 2009 Instructor: B.Bagchi

Maximum Marks 100

Throughout, Ω is an open subset of $\mathbb{R}^d$.

1. A function $f : \Omega \longrightarrow \mathbb{R}$ is said to be homogeneous of degree e (where e is a positive integer) if

   $f(t\ x) = t^e\ f(x)$ for all $t\ \in\ \mathbb{R},\ x \in\ \Omega,\ t\ x\ \in\ \Omega.$

   If such a function is in $C^1(\Omega)$ then show that it satisfies Euler's identity:

   $\sum\limits_{i=1}^{d} x_i(\delta_i f)(x) = e\ f(x), x\ \varepsilon\ \Omega.$ [20]

   (Hint: Differentiate the identity defining homogeneity with respect to t.)

2. Use the change of variable formula to find the value of $\int\limits_{-\infty}^{\infty} e^{-x^2} dx$. [20]

   (Hint: Use polar co-ordinates to compute the square of this integral.)

3. Find the Lebesgue measure of the unit ball (with respect to the Euclidean norm) in $\mathbb{R}^d$. For what value of d it is maximized? [20]

4. If λ_d is the Lebesgue measure on $\mathbb{R}^d, Q$ an open hypercube in $\mathbb{R}^d$ and $\phi : Q \longrightarrow \Omega$ is a diffeomorphism onto Ω show that

$$\lambda_d(\Omega) \leq \int\limits_Q \mid det\ \phi'(x) \mid\ d\ \lambda_d(x).$$

   [20]

5. Use Lagrange's method of multipliers to compute the maximum of $(x_1 \cdots x_d)^2$ subject to the constraint $x_1^2 + \cdots + x_d^2 = 1$. Use the result to deduce that

   $(a_1 \cdots a_d)^{1/d} \leq \frac{a_1+\cdots+a_d}{d}$ for non-negative real numbers $a_1, \cdots, a_d$. [20]